Math 2J
Lecture 15 - 10/31/12

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Recap

- Given a matrix "A", you can sometimes diagonalize it as

$$VAV^{-1} = D$$

- Where "D" is diagonal with eigenvalues on the diagonal and
- "V" is invertible with eigenvectors as the columns.

Further

$$A^p = VD^pV^{-1}$$

$$A^{-1} = VD^{-1}V^{-1}$$

$$det(A) = det(D) = \lambda_1\lambda_2 \ldots \lambda_n$$

Equivalent statements

- "A" is invertible
- "A" is diagonalizable
- "A" has 'n' distinct eigenvectors
- det(A) ≠ 0

$$\det(A) = \lambda_1 \lambda_2 \ldots \lambda_n$$

- $\lambda_i \neq 0$ for all "i"

Special Case

- Eigenvectors belonging to distinct eigenvalues are linearly independent.
- So if all eigenvalues are distinct, "A" is invertible, eigenvectors are distinct, and so forth.
- AND, those eigenvectors don't change direction when multiplied by "A".

Markov Process / Chain

Stochastic Process

- A sequence of experiments for which the outcome depends on chance.

Markov process

- A stochastic process with the following properties
  1. Finite number of outcomes
  2. The outcome of the next experiment only depends on the current state.
  3. Each experiment is the same (i.e. transition probabilities are fixed).

Example

- Economic mobility in a society can be considered as a Markov process. Assume three states: upper, middle, and lower economic classes (indexed by 1, 2 and 3 respectively) and that people can move between these classes. We can encode these probabilities into the following transition matrix:

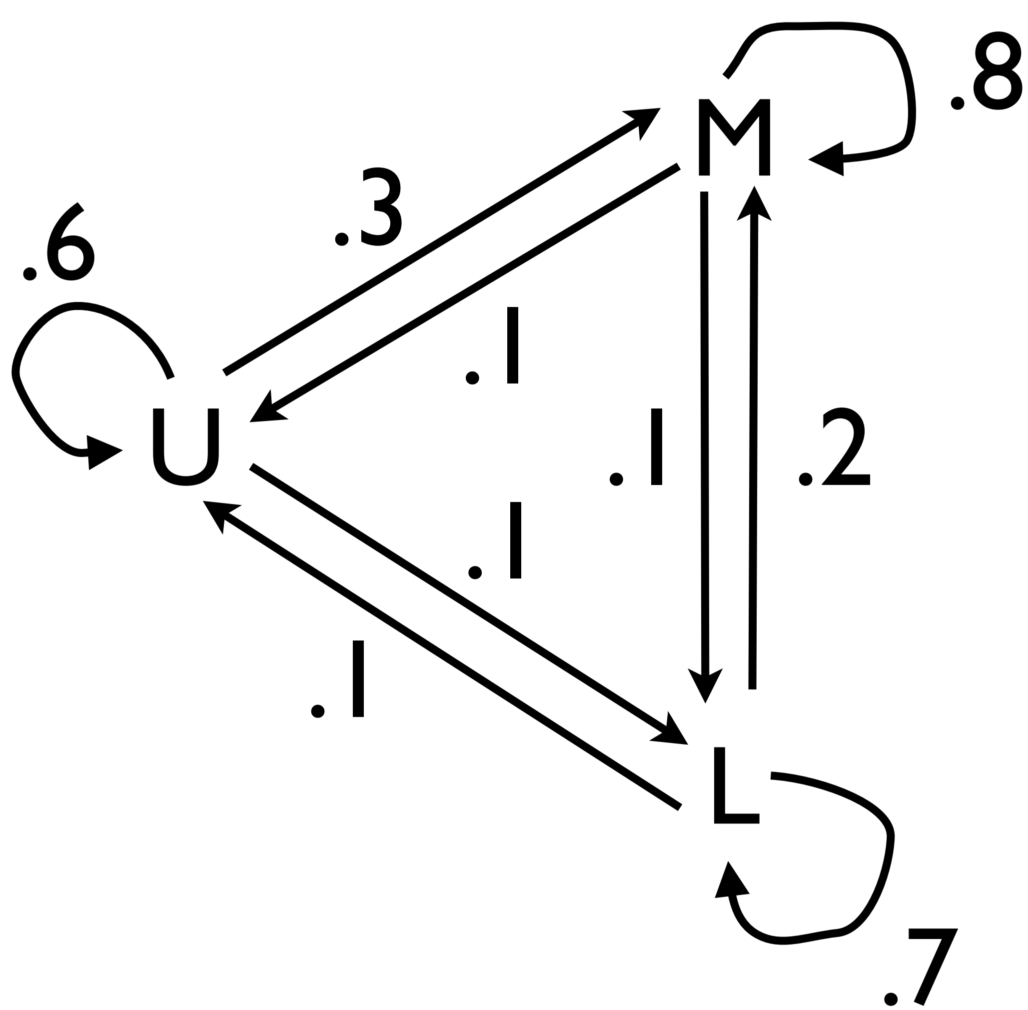
$$T = \begin{array}{c} \begin{array}{ccc} \text{U} & \text{M} & \text{L} \end{array} \\ \begin{bmatrix} .6 & .1 & .1 \\ .3 & .8 & .2 \\ .1 & .1 & .7 \end{bmatrix} \end{array}$$

$$\vec{P}^1 = T\vec{P}^0$$

$$\vec{P}^n = T^n\vec{P}^0$$

- The probability of dropping from upper to middle class is .3.

Schematic



Start

$$\text{End}\quad \begin{array}{c|ccc} & \text{U} & \text{M} & \text{L} \\ \hline \text{U} & .6 & .1 & .1 \\ \text{M} & .3 & .8 & .2 \\ \text{L} & .1 & .1 & .7 \end{array}$$

Markov Process as a Stochastic Matrix

$$T = \begin{bmatrix} .6 & .1 & .1 \\ .3 & .8 & .2 \\ .1 & .1 & .7 \end{bmatrix}$$

- Notice that the columns of "T" each add to 1.
- A **left stochastic matrix** is one who's columns add to 1.
- A **right stochastic matrix** is one who's rows add to 1

Probability Vector

$$\vec{P}^1 = T\vec{P}^0$$
$$\vec{P}^n = T^n\vec{P}^0$$

- "P" represents the state of the system and is called a **probability vector**. It represents the fraction of people in each state and its elements add to "1".

$$\vec{P} = \begin{bmatrix} .1 \\ .6 \\ .3 \end{bmatrix}$$

Properties

- If "T" is a left stochastic matrix and "p" is a probability vector, then the product $T \cdot \vec{p}$ is a probability vector.

Properties

- $\lambda = 1$ is an eigenvalue of every stochastic matrix.
- Furthermore, $\lambda = 1$ is the largest eigenvalue of every stochastic matrix.

Example

- The google page rank matrix is left stochastic!

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \\ P_8 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 0 \\ 1/2 & 0 & 1/2 & 1/3 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/3 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1/3 & 1/3 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1/3 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1/3 & 1 & 1/3 & 0 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \\ P_8 \end{bmatrix}$$

$$\vec{P} = A\vec{P}$$